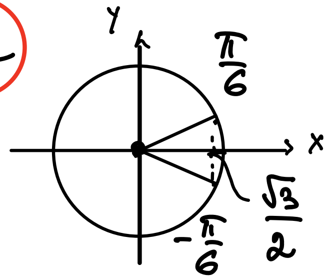
Lösningsskiss till tentamen i Matematisk analys, del 1.

2021 – 10 – 29

1a



$$\cos\left(2x+\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2} \iff 2x+\frac{\pi}{6}=\pm\frac{\pi}{6}+2\pi n$$

$$\iff 2x = 2\pi n \quad \text{eller} \quad 2x = -\frac{\pi}{3}+2\pi n$$

Svar: $x=\pi n,\ x=-\frac{\pi}{6}+\pi n,\quad n\in\mathbb{Z}$.

1b

$$\cos^2 x + 4\sin x - 4 = 0 \quad / \cos^2 x = 1-\sin^2 x /$$

$$1-\sin^2 x + 4\sin x - 4 = 0 \iff \sin^2 x - 4\sin x + 3 = 0$$

$$\left/ \begin{array}{l} t=\sin x \\ -1\le t\le 1 \end{array} \right/ \quad t^2-4t+3=0 \iff (t-1)(t-3)=0 \iff$$

$t_1 = 1$

$t_2 = 3$ – falsk rot ty $t_2 \notin [-1,1]$.

$$\sin x = 1 \iff x=\frac{\pi}{2}+2\pi n,\quad n\in\mathbb{Z}$$

Svar: $x=\frac{\pi}{2}+2\pi n,\ n\in\mathbb{Z}$

1c

$$\ln x - \ln(3x+2) + \ln(x+4) = 0 \qquad (*)$$

Krav: $\begin{cases} x>0 \\ 3x+2>0 \\ x+4>0 \end{cases} \Rightarrow \begin{cases} x>0 \\ x>-2/3 \\ x>-4 \end{cases} \Rightarrow x>0$

$$(*) \iff \ln x + \ln(x+4) = \ln(3x+2) \iff \ln x(x+4) = \ln(3x+2)$$

$$/\ln \text{ är injektiv}/ \iff x(x+4) = 3x+2 \iff x^2+x-2=0$$

$\iff (x-1)(x+2)=0$ dvs $x=1$ eller $x=-2$ – falsk rot

Svar: $x=1$

2 För att bestämma V_f behöver vi undersöka

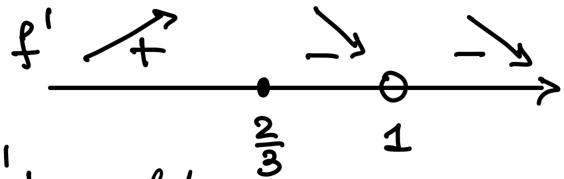
$$f(x)=\frac{e^{-3x}}{x-1}.$$

- $D_f = \{x\in\mathbb{R} :\ x\neq 1\}$
- $\lim\limits_{x\to-\infty} f(x) = -\infty,\quad \lim\limits_{x\to 1^-} f(x) = -\infty,\quad \lim\limits_{x\to 1^+} f(x) = \infty$

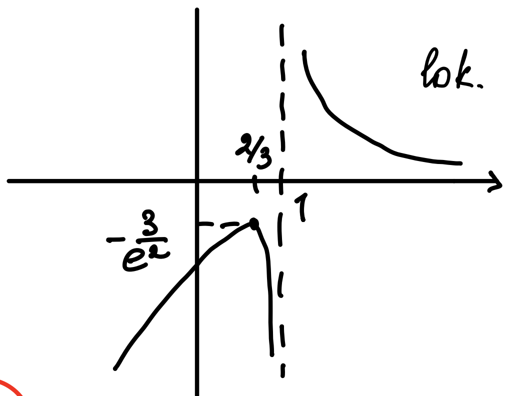
  $\lim\limits_{x\to\infty} f(x) = 0^+.$

- $f'(x) = \dfrac{-3e^{-3x}(x-1) - e^{-3x}}{(x-1)^2} = \dfrac{-e^{-3x}(3x-2)}{(x-1)^2} = 0 \iff$

$x = \frac{2}{3}$



$f\left(\frac{2}{3}\right) = \dfrac{e^{-2}}{-1/3} = -\dfrac{3}{e^2}$.



Svar: $V_f = \left]-\infty, -\frac{3}{e^2}\right] \cup \left]0, \infty\right[$

3a

$$\lim_{x\to 1} \frac{x^2+2x-3}{x^2-3x+2} = \lim_{x\to 1} \frac{(x-1)(x+3)}{(x-1)(x-2)} = -4.$$

3b

$$\lim_{x\to\infty}\left(\sqrt{x^2+x+1} - x\right) = \lim_{x\to\infty} \frac{\cancel{x^2}+x+1-\cancel{x^2}}{\sqrt{x^2+x+1}+x} =$$

$$= \lim_{x\to\infty} \frac{x\left(1+\frac{1}{x}\right)}{x\left(\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+1\right)} = \frac{1}{2}.$$

3c

$$\lim_{x\to 1} \frac{e^{x-1}-1}{\sin(4x-4)} = \Big/ \begin{array}{c} t = x-1 \\ t\to 0 \\ \text{då } x\to 1 \end{array} \Big/ = \lim_{t\to 0} \frac{e^t-1}{\sin 4t} =$$

$$= \lim_{t\to 0} \underbrace{\frac{e^t-1}{t}}_{\to 1} \cdot \frac{t}{4t} \cdot \underbrace{\frac{1}{\frac{\sin 4t}{4t}}}_{\to 1} = \frac{1}{4}.$$

Svar: a) -4 b) $\frac{1}{2}$ c) $\frac{1}{4}$.

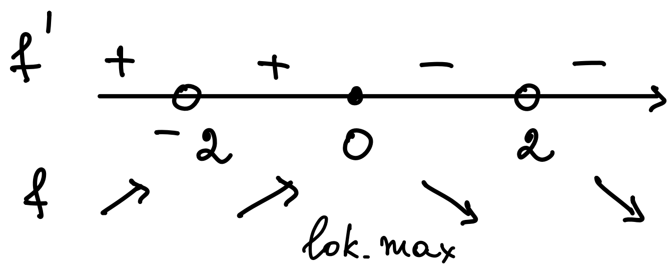
4 $f(x) = \dfrac{x^2}{x^2-4}$. • $D_f = \{x \in \mathbb{R} : x \neq \pm 2\}$

- $\lim_{x\to -\infty} f(x) = \lim_{x\to\infty} f(x) = 1 \Rightarrow y = 1$ – vågrät asymptot

$\lim_{x\to -2^-} f(x) = \lim_{x\to 2^+} f(x) = \infty$; $\lim_{x\to -2^+} f(x) = \lim_{x\to 2^-} f(x) = -\infty$.

$\Rightarrow \quad x=-2, \quad x=2$ – lodräta asymptoter.

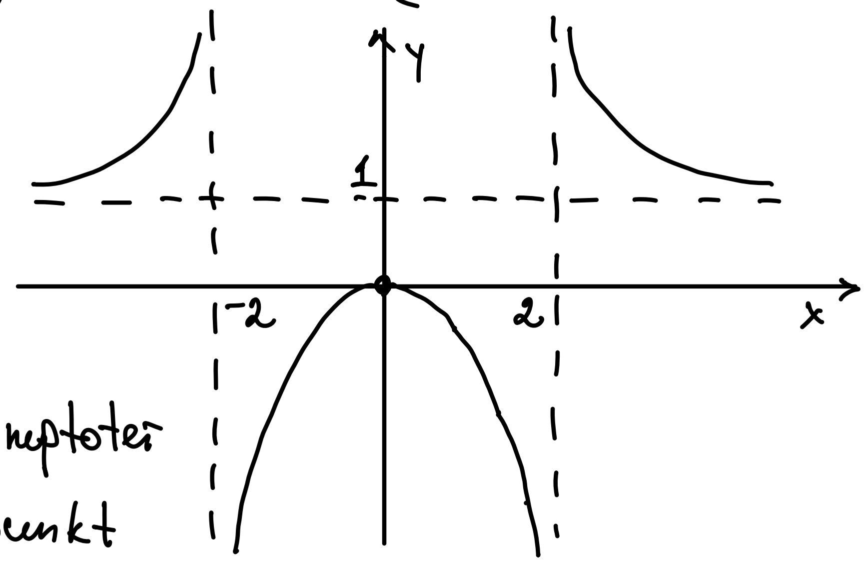
• $f'(x) = \dfrac{2x(x^2-4) - 2x\cdot x^2}{(x^2-4)^2} = \dfrac{-8x}{(x^2-4)^2} = 0 \Leftrightarrow x=0$



$f(0)=0$

Svar: $y=1, \quad x=\pm 2$ - asymptoter

$x=0$ - lok. maximipunkt



5. $z^5 - z^4 + 16z - 16 = 0 \Leftrightarrow z^4(z-1) + 16(z-1) = 0$

$\Leftrightarrow (z^4+16)(z-1) = 0 \Leftrightarrow z_1 = 1$ eller $z^4 = -16$.

Antag att $z = re^{i\varphi}$. Notera att $-16 = 16e^{i\pi} \Rightarrow$

$z^4 = -16 \Leftrightarrow r^4 e^{i4\varphi} = 16e^{i\pi} \Leftrightarrow \begin{cases} r^4 = 16 \\ 4\varphi = \pi + 2\pi n, \end{cases}$

$n = 0, 1, 2, 3.$

$\Leftrightarrow \begin{array}{l} r = 2 \\ \varphi = \dfrac{\pi + 2\pi n}{4}, \quad n = 0,1,2,3. \end{array}$

$n=0 \Rightarrow z_2 = 2e^{i\frac{\pi}{4}} = 2(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) = \sqrt{2} + i\sqrt{2}$

$n=1 \Rightarrow z_3 = 2e^{i\frac{3}{4}\pi} = 2(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi) = -\sqrt{2} + i\sqrt{2}$

$n=2 \Rightarrow z_4 = 2e^{i\frac{5}{4}\pi} = 2(\cos\frac{5}{4}\pi + i\sin\frac{5}{4}\pi) = -\sqrt{2} - i\sqrt{2}$

$n=3 \Rightarrow z_5 = 2e^{i\frac{7}{4}\pi} = 2(\cos\frac{7}{4}\pi + i\sin\frac{7}{4}\pi) = \sqrt{2} - i\sqrt{2}.$

Svar: $z_1 = 1, \quad z_{2,5} = \sqrt{2} \pm i\sqrt{2}, \quad z_{3,4} = -\sqrt{2} \pm i\sqrt{2}.$

6. $f(x) = \begin{cases} e^{3x} + x, & x \geq 0 \\ ax + b, & x < 0 \end{cases}$ för alla $x \neq 0$ är f deriverbar.

Vi undersöker punkten $x=0$. Notera att deriverbarhet medför kontinuitet, dvs konstanterna a och b måste väljas så att f blir kontinuerlig i punkten $x=0$. f är kontinuerlig i $x=0$ då och endast då $\lim_{x\to 0} f(x) = f(0) \iff$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0) \iff$$

$$1 = \lim_{x\to 0^+} (e^{3x}+x) = \lim_{x\to 0^-} (ax+b) = b \Rightarrow b=1.$$

f är deriverbar i punkten $x=0 \iff f'_-(0)=f'_+(0)$

Eftersom $f'(x) = \begin{cases} 3e^{3x}+1, & x>0 \\ a, & x<0 \end{cases}$ får vi

$$a=f'_-(0) = f'_+(0) = 3+1 \Rightarrow a=4.$$

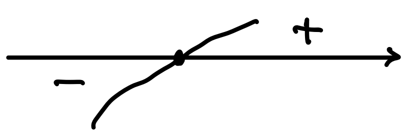
Svar: $a=4$, $b=1$.

7 $\arctan x \geq \frac{x}{x^2+1} \iff \arctan x - \frac{x}{x^2+1} \geq 0$

Sätt $f(x) = \arctan x - \frac{x}{x^2+1}$; $D_f = \mathbb{R}$

$$f'(x) = \frac{1}{1+x^2} - \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{x^2+1-(1-x^2)}{(x^2+1)^2} = \frac{2x^2}{(x^2+1)^2}$$

$f'(x) \geq 0 \Rightarrow f(x)$ är växande samt $f(0)=0$



Alltså $f(x) \geq 0 \iff x \geq 0$.

Svar: olikheten gäller för $x \geq 0$